J. Fluid Mech. (1985), *vol.* 156, *pp.* 349–367
Printed in Great Britain

Diffraction by a slender ship: uniform theory for head and oblique seas

By J. MARTIN
Department of Mathematics, University of Edinburgh

AND D. BARRIE
Department of Ship and Marine Technology, University of Strathclyde

(Received 7 August 1984)

The diffraction problem for a slender ship held fixed in short regular incident waves is solved by a matched-asymptotic-expansion method which is uniform with respect to all incident wave directions including head seas. Special inner solutions are employed which satisfy the Helmholtz equation in cross-sectional planes of the ship and are non-singular in head seas. The inner expansion of the outer solution is derived directly for all wave directions rather than as a composite. The case of a ship with a long parallel middle body is studied by means of a mixed numerical and analytical solution which explicitly exhibits the transition between the distinctive behaviours of head and oblique seas and distinguishes effects generated at the bow from those of the parallel middle body. Calculations of the pressure distributions and wave elevations along a ship are reported and compared with experimental measurements. The agreement between theory and experiment is generally good, especially at the upwave end of the ship and along the upwave side.

1. Introduction

This paper concerns the diffraction problem of linearized ship hydrodynamics. The ship is held fixed in incident sinusoidal waves, and a time-periodic solution is sought for the velocity potential of the diffracted wave field. From this may be calculated the diffraction pressure on the hull, as required, for instance, in seakeeping calculations. Although direct numerical methods exist for this three-dimensional problem, there are advantages in approximating further by means of a high-frequency slender-body assumption. An asymptotic solution is obtained by matching an outer solution generated by a line of sources and dipoles to a family of two-dimensional inner solutions in cross-sectional planes of the ship. This approximation both reduces computation time and provides insight into the physical processes involved. Special difficulties arise when the waves are incident from ahead or astern or nearly so.

The case of oblique seas, studied by Troesch (1979), yields a 'pure strip theory' (the field near the ship is determined entirely by the local cross-section and the incident wave), but this is singular in the limit of head or stern seas. For the case of exact head seas, studied by Faltinsen (1972) and several subsequent authors, the wave field near the ship has an accumulated 'memory' of the shape of cross-sections previously encountered by the wave. This material will be reviewed briefly in §2 to introduce notation; a more detailed review may be found in Ogilvie (1977), and further applications are discussed by Skjørdal & Faltinsen (1980) and Beck & Troesch (1980).

A uniform treatment covering head and oblique seas was given by Liapis & Faltinsen (1980), using a composite outer solution constructed from those of Troesch and Faltinsen. The present paper has a similar aim but derives a uniform inner expansion of the outer solution directly (§3 and Appendix A) rather than as a composite. It also employs certain special inner solutions for oblique seas (§4 and Appendix B) which violate the usual radiation condition but are non-singular in the head sea limit. The matching is carried out in §5, where a Volterra integral equation is set up for the line source distribution. In recent work by Sclavounos (1981, 1984) similar results are derived using 'unified theory' (a composite of high- and low-frequency slender-body theories). A Volterra integral equation asymptotically equivalent to the present one is obtained as a high-frequency approximation. (This work came to the authors' attention after initial submission of the present paper.) In §6 the case of a ship with a long parallel middle body and relatively short bow section is considered. A mixed numerical and analytical solution of the Volterra integral equation is presented. This explicitly describes the transition between oblique- and head-sea behaviours for waves travelling along the parallel middle body. It also shows that effects generated at the bow decay rapidly along the ship at all wave headings. In the case of head seas, these findings support those of Ursell (1975) for certain model problems involving a half-immersed circular cylinder, treated by a Fourier transform along the cylinder and multipole expansion methods. Numerical results are presented in §7 for pressure distributions and wave elevations along the ore-carrier and the other test shape used in previous experimental and theoretical work. The results suggest that three-dimensional effects in oblique seas are successfully described.

2. Formulation and review

In equilibrium conditions let D denote the fluid domain (assumed to be of infinite depth), F the free surface and S the wetted surface of the hull. Let $Oxyz$ be a set of orthogonal axes with Oxy in the free surface, O at the bow, Ox directed towards the stern and Oz vertically upward. Suppose now that waves are incident upon the ship in a direction making an angle β with Ox. The fluid is assumed to be inviscid, incompressible and the flow irrotational. The usual form of the diffraction problem is then as follows. The velocity potential for the complete wave field is Re $[\phi(x, y, z)\, e^{i\omega t}]$, where

$$\begin{aligned}
&\nabla^2\phi = 0 \quad \text{in } D,\\
&\phi_n = 0 \quad \text{on } S \quad \text{(suffix } n \text{ denotes normal differentiation)},\\
&K\phi - \phi_z = 0 \quad \text{on } F,\\
&\phi \to 0 \quad \text{as } z \to -\infty,\\
&\phi = \phi^{\mathrm{I}} + \phi^{\mathrm{D}},\\
&\phi^{\mathrm{I}} = \exp\left[Kz - \mathrm{i}K(x\cos\beta + y\sin\beta)\right],\\
&\phi^{\mathrm{D}} \text{ represents outgoing waves at infinity},\\
&K = \omega^2/g.
\end{aligned}$$

As usual in slender-body theory, a small parameter ϵ is chosen as a measure of the ratio of transverse to longitudinal lengthscales of the ship (it is helpful but possibly inaccurate to think of $\epsilon = B/L$ = beam/length). In the high-frequency theory the wavelength is taken to be $O(\epsilon)\,L$. Let $\nu = K\cos\beta$ and $\mu = K\sin\beta$ denote wavenumbers parallel and perpendicular to the axis of the ship. Initially, oblique seas are considered (i.e. the waves come from a direction not close to head, stern or beam) and so K, ν and μ are all large, of order $(\epsilon L)^{-1}$. In the near field, defined by $0 < x < L$, $(y^2+z^2)^{\frac{1}{2}} = O(\epsilon) L$, the slenderness assumption takes the form of assuming a slow modulation of the incident wave, given by

$$\phi(x,y,z) = \psi(x,y,z)\, \mathrm{e}^{-\mathrm{i}\nu x},$$

where
$$\psi_y, \psi_z = \frac{O(\epsilon^{-1})}{L}, \quad \psi_x = \frac{O(1)}{L}.$$

At station x let $\Delta(x), \Sigma(x)$ and $\mathscr{F}(x)$ denote the cross sections of D, S and F respectively, and let N denote the two-dimensional unit normal to $\Sigma(x)$ directed into the fluid. Then to leading order the problem satisfied by ψ is the following.

Problem $\mathscr{P}_\beta(x)$:

$$\begin{aligned} &\psi_{yy} + \psi_{zz} - \nu^2 \psi = 0 \quad \text{in } \Delta(x), \\ &\psi_N = 0 \quad \text{on } \Sigma(x), \\ &K\psi - \psi_z = 0 \quad \text{on } \mathscr{F}(x), \\ &\psi \to 0 \quad \text{as } z \to -\infty, \end{aligned}$$

and a matching condition as $|y| \to \infty$. At each station x this is a two-dimensional problem, and x-dependence enters only via the shape of $\Sigma(x)$ and the matching.

The far-field solution for the diffracted potential ϕ^{D} must comprise a combination of outgoing waves of sufficient generality to match with the inner solutions. It is customary to start with the potentials of a line source distribution along $y = z = 0$, $0 \leqslant x \leqslant L$ of unknown density $\sigma(x)\,\mathrm{e}^{-\mathrm{i}\nu x}$, where σ is slowly varying, and a similar transverse dipole distribution $\tau(x)\,\mathrm{e}^{-\mathrm{i}\nu x}$. These potentials are first approximated in the far field by certain pole contributions (thus it is really these contributions which provide the ansatz for the outer solution). For matching purposes, inner expansions are then required. In oblique seas these are

$$\sigma(x)\frac{\mathrm{i}K}{\mu}\mathrm{e}^{Kz - \mathrm{i}\nu x - \mathrm{i}\mu|y|} \tag{1}$$

and
$$-\tau(x)\ \mathrm{sgn}\,(y)\ \mathrm{e}^{Kz - \mathrm{i}\nu x - \mathrm{i}\mu|y|} \tag{2}$$

for the source and dipole potentials respectively, and lead to the predictable matching condition to be imposed on the inner problems:

$$\psi \sim \mathrm{e}^{Kz - \mathrm{i}\mu y} + c_{\pm}(x)\ \mathrm{e}^{Kz - \mathrm{i}\mu|y|} \quad \text{as } y \to \pm\infty, \tag{3}$$

where $c_{\pm} = \mp\tau + \mathrm{i}K\sigma/\mu$. $c_{\pm}(x)$ are determined entirely by the inner problem $\mathscr{P}_\beta(x)$ (so that it is a 'pure strip theory'), and then σ and τ may be inferred. The factor $1/\mu$ multiplying the source term suggests difficulties in the head-sea limit, and this is confirmed by properties of the inner problems.

Each oblique-sea inner problem, with matching condition (3), may be solved by reformulating it as an integral equation over Σ using the Green function

$$\gamma(y,z,\eta,\zeta) = K_0(\nu\rho) + K_0(\nu\rho') + \int_\Gamma \frac{K}{\nu\cosh w - K}\exp\left[\mathrm{i}\nu\,|y-\eta|\sinh w + \nu(z+\zeta)\cosh w\right]\mathrm{d}w, \tag{4}$$

where K_0 is a modified Bessel function, $\rho = \{(y-\eta)^2 + (z-\zeta)^2\}^{\frac{1}{2}}$, $\rho' = \{(y-\eta)^2 + (z+\zeta)^2\}^{\frac{1}{2}}$ and Γ denotes a contour consisting of the real axis from $-\infty$ to ∞ indented below the pole on the negative part and above the pole on the positive part. This represents outgoing waves at infinity:

$$\gamma(y,z,\eta,\zeta) - \frac{2\pi\mathrm{i}K}{\mu}\,\mathrm{e}^{K(z+\zeta)-\mathrm{i}\mu|y-\eta|} \to 0 \quad \text{as } |y-\eta| \to \infty. \tag{5}$$

It is evident from (4) and (5) that γ is singular in the head-sea limit ($\mu \to 0$) and consequently a solution of problem $\mathscr{P}_\beta(x)$ obtained in this way is expected to be singular also as $\beta \to 0$. The singularity, in each case, is in the symmetric part (in y) of the function concerned.

For the exact head-sea case, the inner problem is $\mathscr{P}_0(x)$ (i.e. $\nu = K, \mu = 0$). The formal analogue of the radiation condition (3) would be $\psi \sim c(x)\,\mathrm{e}^{Kz}$ as $|y| \to \infty$, but this is not appropriate since Ursell (1968*b*) has shown that, at least for smooth cross-sections with vertical tangents at the free surface, and except possibly at a discrete set of values of K, the problem $\mathscr{P}_0(x)$ has no non-trivial bounded solution. If Σ were independent of x then this problem would describe propagation of regular waves parallel to a uniform cylinder. Ursell's result shows that no physically acceptable solution exists and so waves cannot propagate in this manner without progressive change of form. A Green function can be constructed for this case by a different choice of integration contour:

$$G(y,z,\eta,\zeta) = K_0(K\rho) + K_0(K\rho') + \frac{1}{2}\left(\int_{\Gamma_1} + \int_{\Gamma_2}\right)\frac{1}{\cosh w - 1} \times \exp\left[\mathrm{i}K|y-\eta|\sinh w + K(z+\zeta)\cosh w\right]\mathrm{d}w, \tag{6}$$

where Γ_1 and Γ_2 denote contours consisting of the real axis from $-\infty$ to $+\infty$ indented respectively below and above the double pole at $w = 0$. This Green function has unusual behaviour at infinity:

$$G(y,z,\eta,\zeta) + 2\pi K|y-\eta|\,\mathrm{e}^{K(z+\zeta)} \to 0 \quad \text{as } |y-\eta| \to \infty,$$

which indicates the growth rate at infinity to be expected for non-trivial solutions of problem $\mathscr{P}_0$. Ursell (1968*b*) has shown that, for cross-sections restricted as before, and except possibly at a discrete set of values of K, specification of the linear growth rate leads to an existence and uniqueness result as follows. For each cross-section $\Sigma(x)$ there is a unique coefficient $B_*(x)$ such that the problem $\mathscr{P}_0(x)$ has a unique solution ψ_* satisfying the condition

$$\psi_* - (1 + B_* K|y|)\,\mathrm{e}^{Kz} \to 0 \quad \text{as } |y| \to \infty. \tag{7}$$

B_* turns out to be an important coefficient, which carries information of the section shape into the far field.

In head seas the outer solution plays a more significant role. By symmetry, only the source distribution is required for the diffracted wave. The inner expansion of the outer diffraction potential is

$$\phi^{\mathrm{D}}(x,y,z) \sim \mathrm{e}^{Kz-\mathrm{i}Kx}\{\underbrace{\mathrm{V}_0\,\sigma(x)}_{O(\epsilon^{-\frac{1}{2}}\sigma)} + \underbrace{\sigma(x)\,K|y|}_{O(\sigma)}\} + o(\sigma) \tag{8}$$

as $\epsilon \to 0$, with $Ky = O(1)$, where V_0 denotes the Abel operator

$$\mathrm{V}_0\,\sigma(x) = -\left(\frac{K}{2\pi\mathrm{i}}\right)^{\frac{1}{2}}\int_0^x \frac{\sigma(\xi)\,\mathrm{d}\xi}{(x-\xi)^{\frac{1}{2}}},$$

and the indicated orders apply well away from the ends of the ship. ($i^{\frac{1}{2}}$ is always taken to denote $e^{+\frac{1}{4}i\pi}$.) This result was derived by Faltinsen (1972), but with an additional term. A generalization of (8) is derived in Appendix A, where it is also explained why the additional term should not appear. An inner solution of the form

$$\phi(x,y,z) = p(x)\,\psi_*(x,y,z)\,e^{-iKx}$$

is to be matched to the outer using (7) and (8). The method proposed by Maruo & Sasaki (1974) is to match separately the terms independent of y and terms linear in $|y|$, giving respectively

$$p(x) = 1+V_0\,\sigma(x)$$

and

$$p(x)\,B_*(x) = \sigma(x).$$

It may be noted that this matching mixes terms of different orders in (8). Mei & Tuck (1980) argue that, while this is acceptable (but of no particular virtue) on the middle body of the ship, it actually improves the order of the error near the bow (as compared with a matching order by order in (8)). Elimination of p then gives the Volterra integral equation

$$\sigma = B_*(1+V_0\,\sigma).$$

An alternative treatment by Haren & Mei (1981), for the head-sea case, replaces the source distribution for the outer solution by a parabolic approximation in an intermediate region. This approximation has the physical interpretation of allowing waves near the ship to propagate only from the bow towards the stern. Such a restriction is consistent with the previous approach provided that the additional term does not appear in (8). Waves travelling from stern to bow appear only at higher order.

The uniform treatment of head and oblique seas by Liapis & Faltinsen (1980) begins by considering nearly head seas, $\beta = O(\epsilon)$, and presents an inner expansion of the outer solution analogous to (8) with the additional term. A composite of this expansion and the oblique-sea result (1) is then observed. This provides a smooth transition from oblique seas to head seas, though some doubt remains as to its validity when β is small but not $O(\epsilon)$. In oblique seas, the composite as it stands (i.e. without reapproximating by (1)) is not expressed in terms of the appropriate travelling-wave solutions required for matching by analytical means. Liapis & Faltinsen therefore present a numerical method for solving inner problems which can handle matching conditions of a more general kind. In the present work the matching can be performed in a natural way for all wave headings, and this difficulty is avoided.

3. The outer solution

In the outer region the diffracted potential ϕ^D is taken to be the far-field potential of a line of sources and transverse dipoles in the mean free surface along the centreline of the ship. For a source distribution of line density $\sigma(x)$ the appropriate potential is

$$\Phi(\sigma : x,y,z) = \frac{iK}{2\pi}e^{Kz}\int_{-\infty}^{\infty}\frac{\sigma^*(k+\nu)}{F(k)}\,e^{ikx-iF(k)|y|}\,dk,$$

where

$$\sigma^*(k) = \int_0^L \sigma(x)\,e^{-ikx}\,dx$$

and

$$F(k) = \begin{cases} -i(k^2-K^2)^{\frac{1}{2}} & (|k|>K), \\ (K^2-k^2)^{\frac{1}{2}} & (|k|\leqslant K). \end{cases}$$

For a transverse dipole distribution of line density $\tau(x)$ the potential is

$$-\frac{1}{K}\frac{\partial}{\partial y}\Phi(\tau:x,y,z) = -\frac{1}{2\pi}\,\mathrm{e}^{Kz}\,\mathrm{sgn}\,(y)\int_{-\infty}^{\infty}\tau^*(k+\nu)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}F(k)|y|}\,\mathrm{d}k.$$

Inner approximations of these potentials, uniformly valid for oblique and head seas are provided by the following result.

THEOREM 1. *Suppose there are constants C and $n > \frac{3}{2}$ (independent of k, L and β) such that $|\sigma^*(k)| < CL/(1+|kL|^n)$ for all k. Then as $\epsilon \to 0$ with Ky, $Kz = O(1)$, and $0 \leqslant \beta \leqslant \beta_1 < \frac{1}{2}\pi$,*

$$\Phi(\sigma:x,y,z) = \mathrm{e}^{Kz-\mathrm{i}\nu x}\{\mathrm{V}_\beta\,\sigma(x)\cos\mu y + \sigma(x)\frac{K}{\mu}\sin\mu|y|\} + O(\epsilon^{(n-\frac{3}{2})/(2n+1)}) \tag{9}$$

and
$$-\frac{1}{K}\frac{\partial}{\partial y}\Phi(\sigma:x,y,z) = -\sigma(x)\,\mathrm{sgn}\,(y)\,\mathrm{e}^{Kz-\mathrm{i}\nu x-\mathrm{i}\mu|y|} + O(\epsilon^{(n-1)/2n}), \tag{10}$$

where V_β is the Abel operator,

$$\mathrm{V}_\beta\,\sigma(x) = \frac{-K}{[\pi\mathrm{i}(K+\nu)]^{\frac{1}{2}}}\,\mathrm{e}^{-\mathrm{i}(K-\nu)x}\int_0^x \frac{\sigma(\xi)\,\mathrm{e}^{\mathrm{i}(K-\nu)\xi}}{(x-\xi)^{\frac{1}{2}}}\,\mathrm{d}\xi,$$

and the orders of the errors apply uniformly with respect to β.

The proof is summarized in Appendix A and given in detail in Martin (1984). It uses the fact that, for all β, the leading contribution to $\Phi(\sigma:x,y,z)$ as $\epsilon\to 0$ comes from the neighbourhood of the point $k = -\nu$, which, for head or nearly head seas, contains the branch point at $k = -K$. However, in expanding the integrand about $k = -\nu$, it is important to expand only those terms that do not produce divisions by $K-\nu$. The density function σ (and likewise τ) is determined only after matching and so the bound on $|\sigma^*(k)|$ is an *a priori* assumption. It holds with $n = 2$, for instance, if σ is non-zero only for $0 < x < L$, $\sigma(x)\to 0$ as $x \to 0$ or L, and

$$L\{|\sigma'(0)|+|\sigma'(L)|+\int_0^L|\sigma''(x)|\,\mathrm{d}x\}/\sup|\sigma(x)|$$

is bounded uniformly in K, L, β. In the head-sea limit $(\mu\to 0, \nu\to K)$, Theorem 1 reproduces the result (8). In oblique seas, and provided $(K-\nu)x$ is large, the operator V_β may be approximated by an endpoint contribution near $\xi = x$, giving $\mathrm{V}_\beta\sigma \sim \mathrm{i}K\sigma/\mu$, and then (1) is formally obtained.

4. The inner solutions

The inner problem $\mathscr{P}_\beta$ with $\beta \neq 0$ may be solved with an outgoing-wave condition at infinity by use of the Green function (4), but the solution is singular as $\beta\to 0$. The problem $\mathscr{P}_0$ may be solved with the matching condition (7) by use of the Green function (6), which differs from (4) by a different choice of integration contour. A similar change of contour when $\beta \neq 0$ introduces incoming as well as outgoing waves at infinity. Let

$$G(y,z,\eta,\zeta) = K_0(\nu\rho)+K_0(\nu\rho')+\frac{1}{2}\left(\int_{\Gamma_1}+\int_{\Gamma_2}\right)\frac{K}{\nu\cosh w-K}$$
$$\times\exp[\mathrm{i}\nu|y-\eta|\sinh w+\nu(z+\zeta)\cosh w]\,\mathrm{d}w,$$

where Γ_1 and Γ_2 denote contours consisting of the real axis from $-\infty$ to ∞ indented respectively below and above both poles. Then

$$G(y,z,\eta,\zeta)+2\pi\,\mathrm{e}^{K(z+\zeta)}\frac{K}{\mu}\sin\mu\,|y-\eta|\to 0 \quad \text{as } |y-\eta|\to\infty. \tag{11}$$

The intention is to work with inner solutions constructed using this Green function, which is not singular in head seas, and to depend upon the matching with the outer solution to impose the correct radiation condition for all incident wave directions. Since it is known that the singularity in the oblique-sea theory as $\beta\to 0$ occurs in the symmetric part of the problem (in y), symmetric and antisymmetric inner solutions will be constructed separately.

Consider first and ordinary transmission problem for oblique seas, which is to find $\psi = \psi^{\mathrm{T}}$ satisfying $\mathscr{P}_\beta$ with

$$\psi \sim \begin{cases}(\mathrm{e}^{-\mathrm{i}\mu y}+R\,\mathrm{e}^{\mathrm{i}\mu y})\,\mathrm{e}^{Kz} & \text{as } y\to-\infty,\\ T\,\mathrm{e}^{-\mathrm{i}\mu y}\,\mathrm{e}^{Kz} & \text{as } y\to+\infty.\end{cases}$$

The coefficients R and T have to be calculated as part of the solution. The symmetric and antisymmetric parts of ψ^{T} each satisfy the problem $\mathscr{P}_\beta$ and respectively

$$\psi \sim \mathrm{e}^{Kz}\{\tfrac{1}{2}(T+R+1)\cos\mu y+\tfrac{1}{2}\mathrm{i}(T+R-1)\sin\mu\,|y|\},$$
$$\psi \sim \mathrm{e}^{Kz}\{\tfrac{1}{2}(T-R-1)\cos\mu y+\tfrac{1}{2}\mathrm{i}(T-R+1)\sin\mu\,|y|\}\,\mathrm{sgn}\,(y) \quad \text{as } |y|\to\infty.$$

As $\beta\to 0$, R and T are singular, but it is possible to define a new normalization and new constants, suggested by the form of (11), which are not singular. The results are contained in the following theorem, proved in Appendix B.

THEOREM 2. *Let $\Sigma(x)$ be a smooth cross-section with vertical tangents at the free surface. Then for each β, and for each K except possibly a discrete set of values, there are unique coefficients $B_*^{\mathrm{S}}(x)$ and $B_*^{\mathrm{A}}(x)$ such that the problem $\mathscr{P}_\beta(x)$ has unique solutions $\psi_*^{\mathrm{S}}, \psi_*^{\mathrm{A}}$ which are respectively symmetric and antisymmetric in y and satisfy the conditions*

$$\psi_*^{\mathrm{S}}-\mathrm{e}^{Kz}\left\{\cos\mu y+B_*^{\mathrm{S}}(x)\frac{K}{\mu}\sin\mu\,|y|\right\}\to 0$$

$$\psi_*^{\mathrm{A}}-\mathrm{e}^{Kz}\left\{\frac{K}{\mu}\sin\mu y+B_*^{\mathrm{A}}(x)\cos\mu y\,\mathrm{sgn}\,(y)\right\}\to 0 \quad \text{as } |y|\to\infty.$$

B_*^{S} and B_*^{A} depend on x only through the cross-sectional shape Σ, and carry information of this shape into the far field. The proof generalizes that of Ursell (1968*b*) for head seas. For $\beta \neq 0$ the theorem is equivalent to an existence and uniqueness theorem for the ordinary transmission problem through the relationships

$$\left.\begin{aligned} B_*^{\mathrm{S}} &= \frac{\mathrm{i}\mu(T+R-1)}{K(T+R+1)}, \quad B_*^{\mathrm{A}} = \frac{K(T-R-1)}{\mathrm{i}\mu(T-R+1)},\\ \psi^{\mathrm{T}} &= \tfrac{1}{2}(T+R+1)\,\psi_*^{\mathrm{S}}-\tfrac{1}{2}\mathrm{i}\frac{\mu}{K}(T-R+1)\,\psi_*^{\mathrm{A}},\end{aligned}\right\} \tag{12}$$

which also provide numerical comparisons with published results on R and T. The coefficients B_*^{S} and B_*^{A} are real, and this is equivalent to the well-known results $|R|^2+|T|^2 = 1$ (energy conservation) and $R\overline{T}+T\overline{R} = 0$.

The inner solution is taken to be

$$\phi = e^{-i\nu x}\{p(x)\,\psi_*^{S} + q(x)\,\psi_*^{A}\}, \tag{13}$$

where p and q are determined by the matching.

The special solutions ψ_*^{S} and ψ_*^{A} are computed by means of integral equations over the cross-sections Σ. Let Σ_+ denote the part of Σ in $y > 0$, and suppose this is parametrized by arc-length as $(\eta(s), \zeta(s))$. Then

$$\psi_*^{S}(y,z) = e^{Kz}\cos\mu y + K\int_{\Sigma+} P(s')\,\{G(y,z,\eta(s'),\zeta(s')) + G(y,z,-\eta(s'),\zeta(s'))\}\,ds' \tag{14}$$

and

$$\psi_*^{A}(y,z) = \frac{K}{\mu}\,e^{Kz}\sin\mu y + K\int_{\Sigma+} Q(s')\,\{G(y,z,\eta(s'),\zeta(s')) - G(y,z,-\eta(s'),\zeta(s'))\}\,ds', \tag{15}$$

provided that P and Q satisfy the following Fredholm integral equations expressing the rigid-body boundary condition on Σ_+:

$$\pi P(s) - \int_{\Sigma+} P(s')\left\{\frac{\partial G}{\partial N}(s,s'_+) + \frac{\partial G}{\partial N}(s,s'_-)\right\} ds' = \frac{1}{K}\frac{\partial}{\partial N}\,e^{Kz}\cos\mu y\bigg|_{(\eta(s),\,\zeta(s))}, \tag{16}$$

$$\pi Q(s) - \int_{\Sigma+} Q(s')\left\{\frac{\partial G}{\partial N}(s,s'_+) - \frac{\partial G}{\partial N}(s,s'_-)\right\} ds' = \frac{1}{\mu}\frac{\partial}{\partial N}\,e^{Kz}\sin\mu y\bigg|_{(\eta(s),\,\zeta(s))}, \tag{17}$$

where $G(s,s'_\pm)$ stands for $G(\eta(s),\zeta(s),\ \pm\eta(s'),\zeta(s'))$.

Using the asymptotic property (11) of G, it is then found that

$$B_*^{S} = -4\pi K\int_{\Sigma+} P(s')\,e^{K\zeta(s')}\cos\mu\eta(s')\,ds'$$

and

$$B_*^{A} = 4\pi K\int_{\Sigma+} Q(s')\,e^{K\zeta(s')}\frac{K}{\mu}\sin\mu\eta(s')\,ds'.$$

It should be noted that all these equations are non-singular as $\beta \to 0$.

5. Matching

From Theorem 1 the inner expansion of the complete outer potential is

$$\phi \sim e^{Kz-i\nu x}\left\{e^{-i\mu y} + V_\beta\,\sigma(x)\cos\mu y + \sigma(x)\frac{K}{\mu}\sin\mu|y| - \tau(x)\,e^{-i\mu|y|}\,\mathrm{sgn}\,(y)\right\}.$$

From (13) and Theorem 2, the outer expansion of the complete inner potential is

$$\phi \sim e^{Kz-i\nu x}\left\{p(x)\left[\cos\mu y + B_*^{S}(x)\frac{K}{\mu}\sin\mu|y|\right] + q(x)\left[\frac{K}{\mu}\sin\mu y + B_*^{A}(x)\cos\mu y\,\mathrm{sgn}\,(y)\right]\right\}.$$

The matching follows the approach of Maruo & Sasaki (1974), in which all the terms quoted above are matched simultaneously, even though, in head seas, this mixes different orders at stations well away from the bow. Four equations are obtained by matching the terms in cosine and in sine for $y > 0$ and $y < 0$:

$$p+qB_*^{\mathrm{A}} = 1+\mathrm{V}_\beta\sigma-\tau,$$
$$p-qB_*^{\mathrm{A}} = 1+\mathrm{V}_\beta\sigma+\tau,$$
$$pB_*^{\mathrm{S}}+q = -\frac{\mathrm{i}\mu}{K}+\sigma+\frac{\mathrm{i}\mu\tau}{K},$$
$$-pB_*^{\mathrm{S}}+q = -\frac{\mathrm{i}\mu}{K}-\sigma+\frac{\mathrm{i}\mu\tau}{K}.$$

It follows that σ satisfies the Volterra integral equation

$$\sigma = B_*^{\mathrm{S}}(1+\mathrm{V}_\beta\sigma), \tag{18}$$

while τ is given explicitly as

$$\tau = \frac{\mu B_*^{\mathrm{A}}}{\mu B_*^{\mathrm{A}}-\mathrm{i}K}. \tag{19}$$

The coefficients of the inner solution are related to σ and τ by

$$p = \frac{\sigma}{B_*^{\mathrm{S}}}, \quad q = -\frac{\tau}{B_*^{\mathrm{A}}}.$$

After (18) has been solved numerically, all aspects of the inner and outer solutions may be computed.

In oblique seas and provided $(K-\nu)x$ is large, it has been noted that $\mathrm{V}_\beta\sigma \sim \mathrm{i}K\sigma/\mu$. Equation (18) then formally gives

$$\sigma = \frac{\mu B_*^{\mathrm{S}}}{\mu-\mathrm{i}KB_*^{\mathrm{S}}}. \tag{20}$$

Equations (20) and (19) may be shown to agree with the results (1) and (2) obtained directly from oblique-sea theory.

The integral equation (18) is asymptotically equivalent, both for head and oblique seas, to one given by Sclavounos (1981), obtained as a high-frequency approximation of the 'unified theory'. Sclavounos also uses inner solutions equivalent to ψ_*^{S} and ψ_*^{A} and the Green function G, considered as $\mathrm{Re}\,[\gamma]$. Results like Theorems 1 and 2 are used, but proofs are not given in detail.

6. The effect of a long parallel middle body

The integral equation (18) will be studied for a simplified ship-like form consisting of a bow section and then a long parallel middle body of arbitrary cross-section. Consider first the case

$$B_*^{\mathrm{S}}(x) = \begin{cases} b & (x \geqslant l), \\ 0 & (x < l), \end{cases}$$

where b and l are constants. This is a purely theoretical case; the tempting physical interpretation as a uniform semi-infinite cylinder in $x \geqslant l$ clearly violates the slenderness assumption at its end. Let $N = K-\nu$ and $M = b^2K^2/(K+\nu)$, which are found to be wave-numbers associated with the variation of σ along the ship. As a result of slenderness M is large, of order $(\epsilon L)^{-1}$, but N may be large or small. Let U denote the operator defined by

$$\mathrm{U}f(x) = \frac{1}{(\pi\mathrm{i})^{\frac{1}{2}}}\int_l^x \frac{f(\xi)\,\mathrm{d}\xi}{(x-\xi)^{\frac{1}{2}}},$$

which has the property that

$$U^2 f(x) = -\mathrm{i}\int_l^x f(\xi)\,\mathrm{d}\xi. \tag{21}$$

Then (18) may be written as

$$\mathrm{e}^{\mathrm{i}Nx}\sigma(x) = b\,\mathrm{e}^{\mathrm{i}Nx} - M^{\frac{1}{2}}\,U\{\mathrm{e}^{\mathrm{i}Nx}\sigma(x)\}. \tag{22}$$

Equation (22) may be solved explicitly as follows. Operate with U and use (21), eliminate $U\{\mathrm{e}^{\mathrm{i}Nx}\sigma(x)\}$ using (22) again, and solve the elementary integral equation that results. One obtains $\sigma(x) = \sigma_0(x)$, where

$$\sigma_0(x) = \frac{\mu b}{\mu - \mathrm{i}Kb} + \frac{Mb}{N+M}\frac{\mathrm{e}^{-\mathrm{i}N(x-l)}}{[\pi \mathrm{i} M(x-l)]^{\frac{1}{2}}}\{F(M(x-l)) + \overline{F(N(x-l))}\} \tag{23}$$

and

$$F(X) = \mathrm{i}\int_0^\infty \frac{\mathrm{e}^{-t}\,\mathrm{d}t}{(1+\mathrm{i}t/X)^{\frac{1}{2}}}.$$

F is related to a Fresnel integral and has the asymptotic properties

$$F(X) = \mathrm{i} + \tfrac{1}{2}X^{-1} + O(X^{-2}) \quad \text{as } X \to \infty \tag{24}$$

and

$$F(X) = (\pi \mathrm{i} X)^{\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}X} + O(X) \quad \text{as } X \to 0. \tag{25}$$

For the exact head-sea case ($\mu = 0, \nu = K, N = 0$), by (25),

$$\sigma_0(x) = \left[\frac{2}{\pi \mathrm{i} K(x-l)}\right]^{\frac{1}{2}} F(\tfrac{1}{2}b^2 K(x-l)),$$

and many wavelengths from the bow, by (24),

$$\sigma_0(x) = \left(\frac{2\mathrm{i}}{\pi K x}\right)^{\frac{1}{2}} + O(Kx)^{-\frac{3}{2}} \quad \text{as } Kx \to \infty,$$

in agreement with Faltinsen (1972) and Ursell (1975). For the oblique-sea case the full solution (23) is required, but many wavelengths from the bow both appearances of F may be approximated by (24). The leading terms cancel, leaving

$$\sigma_0(x) = \frac{\mu b}{\mu - \mathrm{i}Kb} + \frac{\pi \mathrm{i}\mu}{2K}\,\mathrm{e}^{-\mathrm{i}N(x-l)}\,(\pi \mathrm{i} N x)^{-\frac{3}{2}} + O(Kx)^{-\frac{5}{2}} \tag{26}$$

as $Kx \to \infty$. The first term agrees with the 'pure strip theory' for oblique seas (20). The next term contains an end effect, which decays as the inverse $\frac{3}{2}$ power and thus more rapidly than the inverse $\frac{1}{2}$ power in head seas. The transition between the two is described by (23).

In order to remove the non-physical abrupt change at $x = l$ consider now the case

$$B_*^{\mathrm{S}}(x) = \begin{cases} b & (x \geqslant l), \\ 0 & (x \leqslant 0), \end{cases}$$

with a smooth transition from 0 to b in $0 < x < l$ representing a slender bow section. Suppose the integral equation (18) has been solved numerically or otherwise for $\sigma(x) = s(x)$, $0 \leqslant x \leqslant l$, which will be considered a known function. Let

$$w(x) = \frac{1}{(\pi \mathrm{i})^{\frac{1}{2}}}\int_0^l \frac{\mathrm{e}^{\mathrm{i}N\xi}s(\xi)}{(x-\xi)^{\frac{1}{2}}}\,\mathrm{d}\xi \quad (x \geqslant l).$$

Then (18) becomes

$$\mathrm{e}^{\mathrm{i}Nx}\sigma(x) = b\,\mathrm{e}^{\mathrm{i}Nx} - M^{\frac{1}{2}}\,U\{\mathrm{e}^{\mathrm{i}Nx}\sigma(x)\} - M^{\frac{1}{2}}w(x) \quad (x \geqslant l),$$

which may be solved explicitly as described above to give $\sigma(x) = \sigma_0(x)+\sigma_1(x)$, where σ_0 is given by (23) and

$$\sigma_1(x) = -\mathrm{e}^{-\mathrm{i}(M+N)x} \int_x^{\infty} \mathrm{e}^{\mathrm{i}M\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\{M\,\mathrm{U}w(\xi)-M^{\frac{1}{2}}w(\xi)\}\,\mathrm{d}\xi. \tag{27}$$

The proof of (27) makes use of the result

$$s(l) = b-M^{\frac{1}{2}}w(l)\,\mathrm{e}^{-\mathrm{i}Nl},$$

which is just the integral equation with $x = l$, and the property of U (proved by a Fourier transform) that

$$\int_l^{\infty} \mathrm{e}^{\mathrm{i}M\xi}\{\mathrm{i}M^{\frac{1}{2}}f(\xi)+\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{U}f(\xi)\}\,\mathrm{d}\xi = 0$$

for all $M > 0$ and any Fourier-transformable function f that vanishes for $x < l$.

The behaviour of σ_1 many wavelengths from the bow is of principal interest. As $x/l\to\infty$ one may show that

$$w(x) = \frac{1}{(\pi \mathrm{i}x)^{\frac{1}{2}}}\int_0^l \mathrm{e}^{\mathrm{i}N\xi}s(\xi)\,\mathrm{d}\xi+O\left(\frac{x}{l}\right)^{-\frac{3}{2}}$$

and

$$\mathrm{U}w(x) = \frac{1}{\pi \mathrm{i}}\int_0^l \mathrm{e}^{\mathrm{i}N\xi}\cos^{-1}\left(\frac{2l-x-\xi}{x-\xi}\right)s(\xi)\,\mathrm{d}\xi$$
$$= -\mathrm{i}\int_0^l \mathrm{e}^{\mathrm{i}N\xi}\,s(\xi)\,\mathrm{d}\xi+\frac{2\mathrm{i}}{\pi x^{\frac{1}{2}}}\int_0^l \mathrm{e}^{\mathrm{i}N\xi}(l-\xi)^{\frac{1}{2}}\,s(\xi)\,\mathrm{d}\xi+O\left(\frac{x}{l}\right)^{-1}$$

(the inverse cosine takes values between 0 and π). It follows from (27) that

$$\sigma_1(x) = -A\left(\frac{x}{l}\right)^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}Nx}+O\left(\frac{x}{l}\right)^{-2}$$

as $x/l\to\infty$, where

$$A = \frac{1}{\pi}l^{-\frac{3}{2}}\int_0^l \mathrm{e}^{\mathrm{i}N\xi}(l-\xi)^{\frac{1}{2}}s(\xi)\,\mathrm{d}\xi+\tfrac{1}{2}\mathrm{i}(\pi \mathrm{i}Ml)^{-\frac{1}{2}}\,l^{-1}\int_0^l \mathrm{e}^{\mathrm{i}N\xi}\,s(\xi)\,\mathrm{d}\xi.$$

The influence of a smooth bow section thus decays at least as fast as the inverse $\frac{3}{2}$ power of distance whatever the incident wave direction. This supports the finding of Ursell (1975) for head seas, using a model problem in which a bow was represented by a source distribution on a finite part of an infinite circular cylinder. For head or nearly head seas the first term in A is dominant, since Ml is large for a slender bow. In oblique seas Nl is also large, and then A can be further approximated by endpoint contributions to give

$$\sigma_1(x) \sim -\frac{\pi \mathrm{i}s(l)\,(\mu-\mathrm{i}Kb)}{2Kb}\mathrm{e}^{-\mathrm{i}N(x-l)}\,(\pi \mathrm{i}Nx)^{-\frac{3}{2}}, \tag{28}$$

indicating that the end effect is even smaller.

7. Numerical results

This section describes a direct numerical study of the formulation set out in §§3–5 for some test cases also used in previous work. The Fredholm integral equations (16) and (17) for the special solutions ψ^{S}_{*} and ψ^{A}_{*} (and hence the coefficients B^{S}_{*} and B^{A}_{*}) were solved for a set of cross-sections $\Sigma_+(x)$, spaced along the ship, by a standard collocation method. A check on these computations is provided by (12), which, for

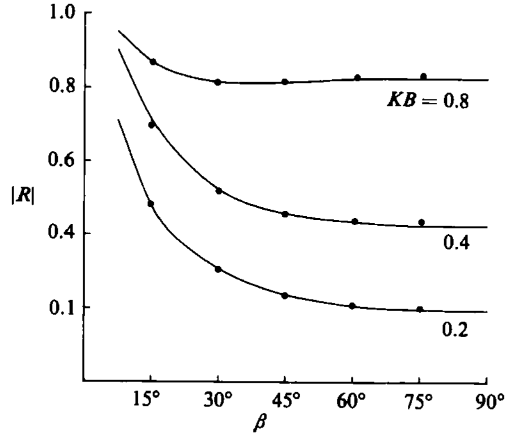

FIGURE 1. Reflection coefficient $|R|$ plotted against heading angle β for a rectangular cross-section (draught: beam = 1:2) at three incident wavelengths given by $KB = 0.2, 0.4, 0.8$, where $B \equiv$ beam and $K \equiv 2\pi/\text{wavelength}$. Comparison of present method (●) and results of Bai (1975) (——).

oblique seas, relate B^S_* and B^A_* to the ordinary reflection and transmission coefficients. Comparisons with results given by Bai (1975) using a finite-element method are shown in figure 1. In this figure $|R|$ is plotted against heading angle, for three wavelengths and a cross-section that is rectangular, with draught: beam = 1:2. Agreement is seen to be good.

The Volterra integral equation (18) for $\sigma(x)$ was solved by a marching procedure, using a modified Simpson rule for the integration. The results agree well with the exact solution (23) for the theoretical case of constant B^S_*.

The complete calculation was performed for the ore-carrier used in previous experimental and theoretical work. This ship has nearly rectangular cross-sections (draught: beam ≈ 2:5) for much of its length (beam: length ≈ 1:6). A body plan is given by Liapis & Faltinsen (1980). The non-dimensional pressure amplitude $|P/\rho gh|$, where $h \equiv$ incident wave amplitude, was computed at points on the hull. Sample results are presented in figures 2 and 3. These figures also include the results of the oblique-sea theory based on (1) and (2), and experimental measurements by Nakamura *et al.* (1973) and by the Society of Ship Research of Japan (1974). Each diagram represents a pressure distribution at a particular cross-section plotted against θ, the polar angle measured from the keel with θ positive on the upwave side. Figure 2 is for heading $\beta = 45°$, wavelength $\lambda = 0.5L$, and includes a forward, midship and aft section (compare Liapis & Faltinsen 1980, figures 3–5; Troesch 1979, figures 5 and 2). Figure 3 is similar, but with $\lambda = L$ (compare Liapis & Faltinsen 1980, figures 9–11). In qualitative terms the findings confirm those of Liapis & Faltinsen; quantitatively the present results may agree slightly better with the measurements, but this is hardly significant. The agreement is found to be remarkably good considering that the ship is not particularly slender and the wavelength not particularly short compared with ship length. Three-dimensional effects, absent from the oblique-sea theory, are most evident on the leeside at the upwave end of the ship (figures $2a, 3a$). Results were also computed with $\beta = 85°$ (nearly beam seas) and $\lambda = \frac{1}{2}L$, but are not included since they do not differ appreciably from those of Liapis & Faltinsen.

An important purpose of a uniform theory is to study to what extent the three-dimensional effects, present in head seas, survive as β is increased from zero.

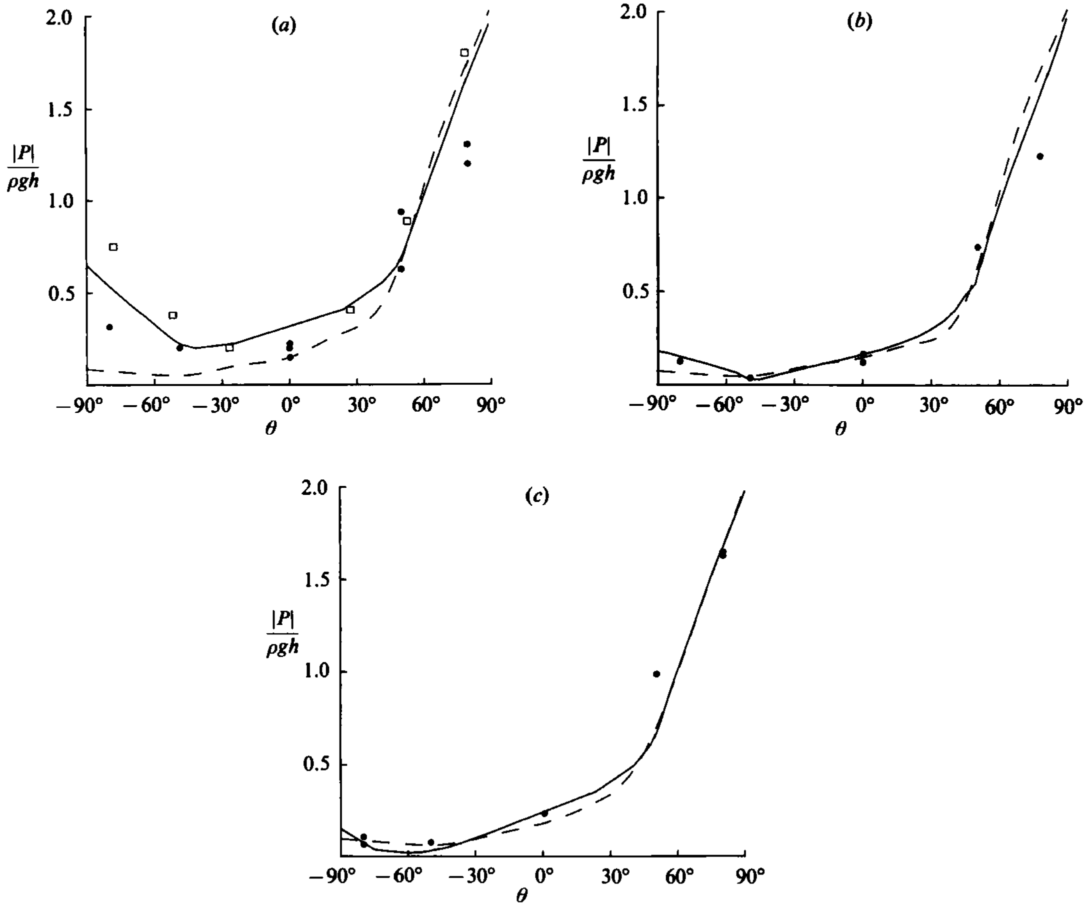

FIGURE 2. Non-dimensional pressure amplitude $|P|/\rho g h$ (where $h \equiv$ incident wave amplitude) on three cross-sections of the hull of an ore-carrier plotted against θ, the polar angle measured from the keel (with θ positive on the upwave side): (*a*) at a forward station $x/L = 0.15$; (*b*) amidships $x/L = 0.5$; (*c*) at an aft station $x/L = 0.75$. Comparison of present method (——), oblique-sea theory (– – –), experimental results of Nakamura *et al.* (1973) (●) and of the Society of Ship Research of Japan (1974) (□). Heading angle $\beta = 45°$ and wavelength $\lambda = 0.5L$.

A convenient indicator of this is the wave amplitude along the side of a ship which is chosen to be symmetrical fore and aft (i.e. about $x = \frac{1}{2}L$). The oblique-sea theory would predict that the wave-amplitude would also be symmetrical about $x = \frac{1}{2}L$, and deviations from symmetry indicate three-dimensional effects. A simplified shiplike form studied by Maruo & Sasaki (1974) was used. It has semicircular cross-sections, uniform parallel middle body and symmetrically tapered ends (beam:length = 3:10, length of tapered ends:length = 1:5). In figure 4 the wave amplitudes along the sides of this form are plotted against x, for $\lambda = 0.5L$ and a range of values of β. Although for the most part the resulting curves become more symmetrical about $x = \frac{1}{2}L$ as β is increased, there is still noticeable non-symmetry even at 50°, indicating the continued presence of three-dimensional effects. However, the non-symmetry is of an order consistent with (26) and (28), indicating that in this example the value of ϵ is too large for the inverse $\frac{3}{2}$ decay of end effects to become fully effective.

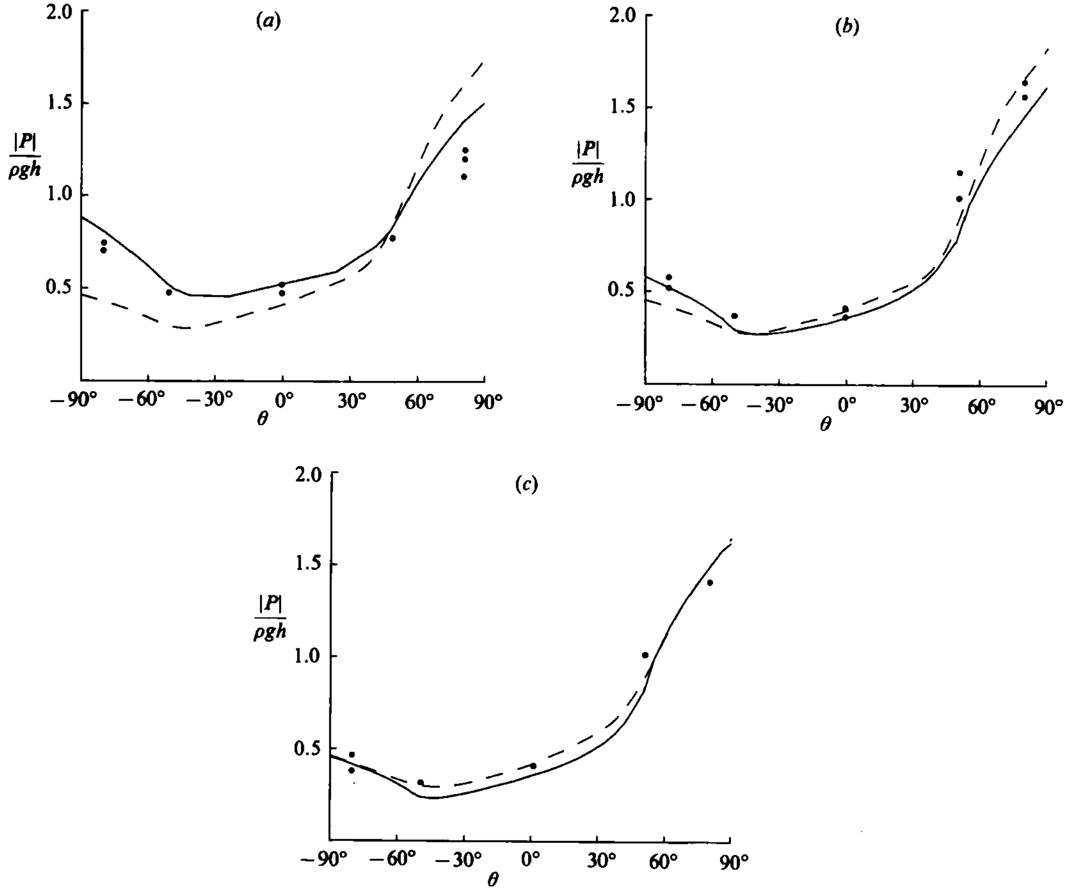

FIGURE 3. As figure 2 with $\lambda = L$.

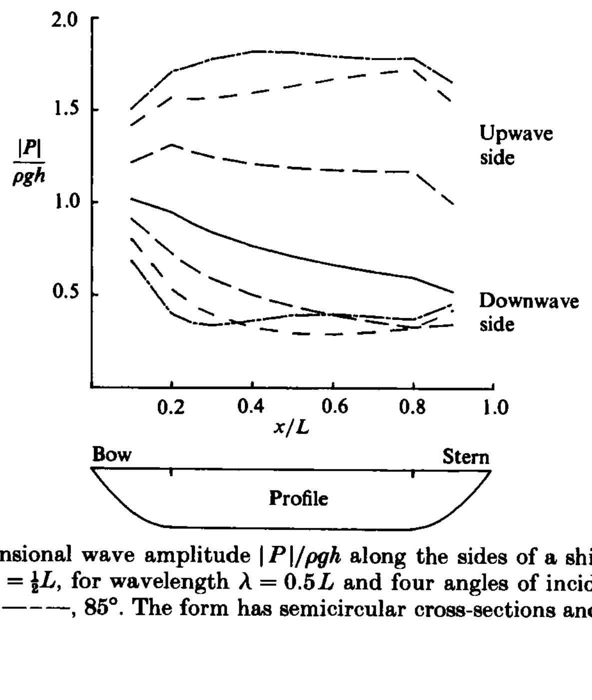

FIGURE 4. Non-dimensional wave amplitude $|P|/\rho g h$ along the sides of a shiplike form which is symmetrical about $x = \frac{1}{2}L$, for wavelength $\lambda = 0.5L$ and four angles of incidence: ——, $\beta = 0°$; — —, 25°; - - -, 50°; — - —, 85°. The form has semicircular cross-sections and the profile shown.

8. Concluding remarks

The diffraction problem for a slender ship in short waves has been solved by a matched-asymptotic-expansion method which is uniform with respect to incident wave direction including head seas. In this method the inner expansion of the outer solution (generated by a line of sources and dipoles) is obtained directly rather than as a composite, and the inner solution is expressed in terms of two special solutions of the Helmholtz equation which are non-singular in the head-sea limit. In oblique seas these solutions represent incoming as well as outgoing waves, but the correct radiation condition is later ensured through the matching. Associated with the special solutions are two coefficients, which depend on the local cross-sectional shape and which, through the matching, carry information of this shape into the outer field. Determination of each coefficient requires the solution of a single Fredholm integral equation over the cross-sectional curve. After matching, the dipole strength is found explicitly and a single Volterra integral equation must be solved to find the source strength. All physical quantities of interest are then easily calculated.

For the case of a ship with a long parallel middle body and relatively short bow section, a mixed numerical and analytical solution of the Volterra integral equation explicitly describes the transition from the 'pure strip theory' results in oblique seas to the inverse square root decay of amplitude with distance along the ship in head seas. It also shows that effects generated at the bow decay as the inverse $\frac{3}{2}$ power of distance along the ship at all wave headings.

When applied to an ore-carrier, the method is found to give good agreement with published experimental results for pressure distributions on the hull, even for waves as long as the ship. For waves incident at 45°, differences between the present results and the oblique-sea pure strip theory are observed, especially near the bow on the down-wave side. The wave amplitude at points along the hull of a shiplike form chosen to be symmetric fore and aft was computed for a range of heading angles. Oblique-sea pure strip theory would predict that this wave amplitude would also be symmetric fore and aft. The present theory for a case with $\lambda/L = 0.5$ shows significant non-symmetry even at incidence angles as large as 50°, indicating the persistence of significant three-dimensional effects.

During the period of this work one of us (D.B.) was employed under the contract Safeship (5) by the U.K. Department of Trade, whose support is gratefully acknowledged.

Appendix A. Summary of proof of Theorem 1

In this summary the detailed derivation of the error bounds is omitted; for a fuller account see Martin (1984). All orders of magnitude are uniform with respect to the heading angle β, $0 \leqslant \beta \leqslant \beta_1 < \frac{1}{2}\pi$. It is assumed that KL and νL are $O(\epsilon^{-1})$, $Ky = O(1)$, but $(K-\nu)L$ may be large or small. The following elementary results are used.

For each $m > 1$ there are constants c_1, c_2, c_3, c_4 such that, for all $\alpha > 0$ and $L > 0$,

$$\int_\alpha^\infty k^{-m}\,|k-\kappa|^{-\frac{1}{2}}\,\mathrm{d}k \begin{cases} < c_1\,\alpha^{-m+1}\,\kappa^{-\frac{1}{2}} & \text{for all } k > 2\alpha, \qquad \text{(A 1)} \\ < c_2\,\alpha^{-m+\frac{1}{2}} & \text{for all } \kappa, \qquad \text{(A 2)} \end{cases}$$

$$\int_{-\alpha}^{\alpha}(1+|kL|^m)^{-1}|k+\kappa|^{-\frac{1}{2}}\,dk\begin{cases}<c_3L^{-\frac{1}{2}} & \text{for all } \kappa, & \text{(A 3)}\\ <c_4L^{-1}\kappa^{-\frac{1}{2}} & \text{for all } \kappa>2\alpha. & \text{(A 4)}\end{cases}$$

Let
$$f(k)=\begin{cases}(k-\nu+K)^{\frac{1}{2}} & (k\geqslant\nu-K).\\ -\mathrm{i}(\nu-K-k)^{\frac{1}{2}} & (k<\nu-K),\end{cases}$$

and
$$g(k)=\begin{cases}(K+\nu-k)^{\frac{1}{2}} & (k\leqslant K+\nu)\\ -\mathrm{i}(k-K-\nu)^{\frac{1}{2}} & (k>K+\nu).\end{cases}$$

Then, with a minor change of variable,

$$-2\pi\mathrm{i}\,\mathrm{e}^{-Kz+\mathrm{i}\nu x}\,\Phi(\sigma:x,y,z)=\int_{-\infty}^{\infty}\frac{K\sigma^*(k)}{f(k)\,g(k)}\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}|y|f(k)\,g(k)}\,\mathrm{d}k.$$

Let $\alpha=K\epsilon^{\delta}$, where $0<\delta<1$, and consider separately the intervals $(-\infty,-\alpha)$, $(-\alpha,\alpha)$, (α,∞). The contribution of (α,∞) may be shown to be $O(\epsilon^{n-1-n\delta+\frac{1}{2}\delta})$ by noting that on this interval $f(k)>k^{\frac{1}{2}}$ and $\operatorname{Im}[f(k)\,g(k)]\leqslant 0$, and using (A 1). Likewise the contribution of $(-\infty,-\alpha)$ is $O(\epsilon^{n-1-n\delta+\frac{1}{2}\delta})$ by noting that on this interval $g(k)>K^{\frac{1}{2}}$ and $\operatorname{Im}[f(k)\,g(k)]\leqslant 0$, and using (A 2). On $(-\alpha,\alpha)$ contributions to the integrand of terms in $\sin(|y|fg)$ and $\cos(|y|fg)$ require different treatments. For the former the regularity of $(\sin\theta)/\theta$ at $\theta=0$ enables the integrand to be approximated by its value at $k=0$ with an error bound uniform in β. One may show that

$$\int_{-\alpha}^{\alpha}\frac{K\sigma^*(k)}{f(k)\,g(k)}\mathrm{e}^{\mathrm{i}kx}\sin(|y|f(k)\,g(k))\,\mathrm{d}k$$
$$=\frac{K\sin[|y|(K^2-\nu^2)^{\frac{1}{2}}]}{(K^2-\nu^2)^{\frac{1}{2}}}\int_{-\alpha}^{\alpha}\sigma^*(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k+O(\epsilon^{-\frac{1}{2}+\delta}).\quad\text{(A 5)}$$

A similar step cannot be performed for the cosine term owing to the factor $1/f(k)$, which cannot be approximated for small k uniformly in β. By approximating all the remaining terms by their values at $k=0$, and using (A 3) and (A 4) to simplify the resulting error bounds, one may show that

$$\int_{-\alpha}^{\alpha}\frac{K\sigma^*(k)}{f(k)\,g(k)}\,\mathrm{e}^{\mathrm{i}kx}\cos(|y|f(k)\,g(k))\,\mathrm{d}k$$
$$=\frac{K\cos[|y|(K^2-\nu^2)^{\frac{1}{2}}]}{(K+\nu)^{\frac{1}{2}}}\int_{-\alpha}^{\alpha}\frac{\sigma^*(k)}{f(k)}\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k+O(\epsilon^{-\frac{1}{2}+\delta}).\quad\text{(A 6)}$$

In the approximations (A 5) and (A 6), the domain of integration may be extended to $(-\infty,\infty)$ with errors $O(\epsilon^{n-1-n\delta+\delta})$ for (A 5) and $O(\epsilon^{n-1-n\delta+\frac{1}{2}\delta})$ for (A 6).

Let δ be chosen so that $n-1-n\delta+\frac{1}{2}\delta=-\frac{1}{2}+\delta$, that is $\delta=(n-\frac{1}{2})/(n+\frac{1}{2})$. The error terms are all then $O(\epsilon^{(n-\frac{3}{2})/(2n+1)})$, and hence small provided $n>\frac{3}{2}$. The result (9) follows by the convolution theorem for Fourier transforms, noting that

$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kx}}{f(k)}\,\mathrm{d}k=\frac{2\pi\,\mathrm{e}^{-\mathrm{i}(K-\nu)\,x}\,H(x)}{(\pi\mathrm{i}x)^{\frac{1}{2}}},$$

where H is the unit step function.

Now consider

$$2\pi\,\operatorname{sgn}(y)\,\mathrm{e}^{-Kz+\mathrm{i}\nu x}\frac{1}{K}\frac{\partial}{\partial y}\Phi(\sigma:x,y,z)=\int_{-\infty}^{\infty}\sigma^*(x)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}|y|f(k)\,g(k)}\,\mathrm{d}k,$$

and split the domain of integration into $(-\infty, -\alpha)$, $(-\alpha, \alpha)$, (α, ∞) as before. It is readily seen that the contributions of $(-\infty, -\alpha)$ and (α, ∞) are both $O(\epsilon^{n-1-n\delta+\delta})$. On $(-\alpha, \alpha)$ arguments similar to that leading to (A 5) give

$$\int_{-\alpha}^{\alpha} \sigma^*(x)\, \mathrm{e}^{\mathrm{i}kx-\mathrm{i}|y|f(k)\,g(k)}\, \mathrm{d}k = \mathrm{e}^{-\mathrm{i}|y|(K^2-\nu^2)^{\frac{1}{2}}} \int_{-\alpha}^{\alpha} \sigma^*(k)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k + O(\epsilon^{-\frac{1}{2}+\delta}). \quad \text{(A 7)}$$

Extension of the domain of integration in the approximation (A 7) to $(-\infty, \infty)$ introduces further errors $O(\epsilon^{n-1-n\delta+\delta})$. Let δ be chosen so that $n-1-n\delta+\delta = -\frac{1}{2}+\delta$, that is $\delta = (n-\frac{1}{2})/n$. The error terms are then all $O(\epsilon^{(n-1)/2n})$, and so small provided $n > 1$. The result (10) follows.

The leading terms in these calculations come from a neighbourhood of $k = 0$ that, for head or nearly head seas, contains the branch point at $k = -(K-\nu)$. Faltinsen (1972) considers, in a similar manner, the contribution of the neighbourhood of the branch point at $k = K+\nu$, and this produces the extra term. However, as noted earlier, the domain (α, ∞), which would contain such a neighbourhood, gives an asymptotically smaller contribution. This conclusion follows from the assumed bound on $|\sigma^*(k)|$ used elsewhere in the argument. The larger result found by Faltinsen is entirely attributable to the final extension of the domain to $(-\infty, \infty)$, which then includes the neighbourhood of $k = 0$ where $|\sigma^*(k)|$ is largest. The contribution of the extension is asymptotically larger than that of the original domain, so this step is invalid. A numerical check on the effect of the extra term was performed for the ore-carrier, described in §6, in head seas of wavelengths $0.5L$ and L (see Barrie 1984). Removal of the extra term was found to increase the magnitude of the total potential at points on the hull by at most 5%.

Appendix B. Proof of Theorem 2

The proof follows closely that of Ursell (1968*b*), and only the necessary modifications will be noted. The kernels of the Fredholm integral equations (16) and (17) may be shown to be bounded, as in Ursell's Appendix 1, and are analytic functions both of K and β. Hence their Fredholm determinants are analytic functions of K and β, and so, for each β, can vanish for at most an enumerable set of values of K, unless they vanish identically. However, when $K = \beta = 0$, the Fredholm determinants do not vanish. For the symmetric case this is exactly the result proved by Ursell, and the antisymmetric case is little different. Let $\mathscr{C}$ denote the closed curve consisting of Σ and its reflection in $z = 0$. Let P and Q be extended to $\mathscr{C}$ as functions symmetric in z and respectively symmetric and antisymmetric in y. Then, for $K = \beta = 0$, (16) and (17) become respectively

$$\pi P(s) - \int_{\mathscr{C}} P(s') \frac{\partial}{\partial N} \log \{[\eta(s)-\eta(s')]^2 + [\zeta(s)-\zeta(s')]^2\}^{\frac{1}{2}}\, \mathrm{d}s' = \left.\frac{\partial z}{\partial N}\right|_{(\eta(s),\, \zeta(s))},$$

$$\pi Q(s) - \int_{\mathscr{C}} Q(s') \frac{\partial}{\partial N} \log \{[\eta(s)-\eta(s')]^2 + [\zeta(s)-\zeta(s')]^2\}^{\frac{1}{2}}\, \mathrm{d}s' = \left.\frac{\partial y}{\partial N}\right|_{(\eta(s),\, \zeta(s))}.$$

The same operator occurs on the left of both equations, and it is known to have non-zero Fredholm determinant as noted by Ursell. The general theory of Fredholm integral equations of the second kind now shows that (16) and (17) have unique solutions P and Q for each β, except for at most an enumerable set of values of K. The existence of the solutions ψ_*^{S} and ψ_*^{A} then follows from (14) and (15) by construction. One can, incidentally, show that when $\beta = 0$, and $K \to 0$, $B^{\mathrm{S}}_{*} \sim KB$, where B denotes the beam, and $B^{\mathrm{A}}_{*} \sim K^2 A$, where A denotes the submerged cross-sectional area.

To show the uniqueness of ψ^{S}_{*} and ψ^{A}_{*}, Fredholm integral equations are constructed by means of Green's Theorem as in Ursell (1968*b*, §4). For the symmetric case the Green function

$$G^*(y,z,\eta,\zeta) = G(y,z,\eta,\zeta) + G(y,z,-\eta,\zeta) - 2\,\mathrm{e}^{K\zeta}\cos\mu\eta\; G(y,z,0,0)$$

is used, while for the antisymmetric case

$$G^*(y,z,\eta,\zeta) = G(y,z,\eta,\zeta) - G(y,z,-\eta,\zeta) + 2\,\mathrm{e}^{K\zeta}\frac{\sin\mu\eta}{\mu}\frac{\partial G}{\partial y}(y,z,0,0).$$

These are exponentially small as $|y| \to \infty$ (see Ursell 1968*a*,*b*). The conditions on ψ^{S}_{*} and ψ^{A}_{*} ensure that they are algebraically bounded for all β (in fact bounded unless $\beta = 0$), and this is sufficient to deduce the integral equations

$$\pi\psi^{\mathrm{S}}_{*}(s) - \int_{\Sigma+} \psi^{\mathrm{S}}_{*}(s')\left\{\frac{\partial G}{\partial N'}(s',s_+) + \frac{\partial G}{\partial N'}(s',s_-)\right\}\mathrm{d}s' = -2\mathrm{e}^{K\zeta(s)}\cos\mu\eta(s)\int_{\Sigma+}\psi^{\mathrm{S}}_{*}(s')\frac{\partial G}{\partial N'}(s';0,0)\,\mathrm{d}s'$$

and

$$\pi\psi^{\mathrm{A}}_{*}(s) - \int_{\Sigma+} \psi^{\mathrm{A}}_{*}(s')\left\{\frac{\partial G}{\partial N'}(s',s_+) - \frac{\partial G}{\partial N'}(s',s_-)\right\}\mathrm{d}s' = 2\mathrm{e}^{K\zeta(s)}\frac{\sin\mu\eta(s)}{\mu}\int_{\Sigma+}\psi^{\mathrm{A}}_{*}(s')\frac{\partial^2 G}{\partial N'\,\partial y'}(s',0,0)\,\mathrm{d}s'.$$

The uniqueness argument of Ursell (1968*b*, §4) now goes through, with $\partial G/\partial y$ replacing G in the antisymmetric case.

REFERENCES

BAI, K. J. 1975 Diffraction of oblique waves by an infinite cylinder. *J. Fluid Mech.* **68**, 513–535.

BARRIE, D. A. 1984 Comparison of results using two forms for the outer solution to the diffraction potential in head seas. *Internal Rep., Dept Ship and Marine Tech., Univ. Strathclyde.*

BECK, R. F. & TROESCH, A. W. 1980 Wave diffraction effects in head seas. *Intl Shipbdng Prog.* **27**, 306–315.

FALTINSEN, O. M. 1972 Wave forces on a restrained ship in head sea waves. In *Proc. 9th Symp. on Naval Hydrodynamics, Paris*, pp. 7–42.

HAREN, P. & MEI, C. C. 1981 Head-sea diffraction by a slender raft with application to wave-power absorption. *J. Fluid Mech.* **104**, 505–526.

LIAPIS, N. & FALTINSEN, O. M. 1980 Diffraction of waves around a ship. *J. Ship Res.* **24**, 147–155.

MARTIN, J. 1984 *Internal Rep., Dept Maths, Univ. Edinburgh.*

MARUO, H. & SASAKI, N. 1974 On the wave pressure acting on the surface of an elongated body fixed in head seas. *J. Soc. Nav. Archit. Japan* **136**, 34–42.

MEI, C. C. & TUCK, E. O. 1980 Forward scattering by long thin bodies. *SIAM J. Appl. Maths* **39**, 178–191.

NAKAMURA, S., TAKAGI, M., SAITO, K. & GANNO, M. 1973 Hydrodynamic pressures on a restrained ship in oblique waves. *J. Soc. Nav. Archit. Japan* **133**, 85–100.

OGILVIE, T. F. 1977 Singular-perturbation problems in ship hydrodynamics. *Adv. Appl. Mech.* **17**, 91–188.

SCLAVOUNOS, P. D. 1981 The interaction of an incident wave field with a floating slender body at zero speed. In *Proc. 3rd Intl Conf. on Numerical Ship Hydrodynamics, Paris.*

SCLAVOUNOS, P. D. 1984 The diffraction of free-surface waves by a slender ship. *J. Ship. Res.* **28**, 29–47.

SKJØRDAL, S. O. & FALTINSEN, O. M. 1980 A linear theory of springing. *J. Ship Res.* **24**, 74–84.

SOCIETY OF SHIP RESEARCH OF JAPAN 1974 Measurement of hydrodynamic pressures acting on the hull of a ship which is constrained in the regular waves. *Committee Rep.* SR131, pp. 76–93.

TROESCH, A. W. 1979 The diffraction forces for a ship moving in oblique seas. *J. Ship Res.* **23**, 127–139.

URSELL, F. 1968*a* The expansion of water-wave potentials at great distances. *Proc. Camb. Phil. Soc.* **64**, 811–826.

URSELL, F. 1968*b* On head seas travelling along a horizontal cylinder. *J. Inst. Maths Applics* **4**, 414–427.

URSELL, F. 1975 The refraction of head seas by a long ship. *J. Fluid Mech.* **67**, 689–703.